Scale Invariance of the PNG Droplet and the Airy Process

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Dedicated with admiration to David Ruelle and Yasha Sinai at the occasion of their 65*-th birthday.*

Abstract

We establish that the static height fluctuations of a particular growth model, the PNG droplet, converges upon proper rescaling to a limit process, which we call the Airy process $A(y)$. The Airy process is stationary, it has continuous sample paths, its single "time" (fixed y) distribution is the Tracy-Widom distribution of the largest eigenvalue of a GUE random matrix, and the Airy process has a slow decay of correlations as y^{-2}. Roughly the Airy process describes the last line of Dyson's Brownian motion model for random matrices. Our construction uses a multi-layer version of the PNG model, which can be analyzed through fermionic techniques. Specializing our result to a fixed value of y, one reobtains the celebrated result of Baik, Deift, and Johansson on the length of the longest increasing subsequence of a random permutation.

1 The PNG droplet

The polynuclear growth (PNG) model is a simplified model for layer by layer growth [1, 2]. Initially one has a perfectly flat crystal in contact with its supersaturated vapor. Once in a while a supercritical seed is formed, which then spreads laterally by further attachment of particles at its perimeter sites. Such islands coalesce if they are in the same layer and further islands may be nucleated upon already existing ones. The PNG model ignores the lateral lattice structure and assumes that the islands are circular and spread at constant speed.

In this paper we study the one-dimensional version in the particular geometry where nucleation only above the ground layer $[-t, t]$ is allowed. To be precise: at time t the height is given by the (random) function $h(x, t)$, $x \in \mathbb{R}$. One requires $h(x, t) = 0$ for $|x| > t$, in particular $h(x, 0) = 0$ for $x \in \mathbb{R}$. At a given time t, $h(x, t)$ is piecewise constant, takes nonnegative integer values, and has jumps of size ± 1 only. The jumps are called steps and we distinguish between up-steps (jump size 1) and down-steps (jump size -1). The dynamics has a deterministic piece, according to which down-steps move with velocity 1 and up-steps with velocity -1. Surface steps disappear upon collision. In addition there are nucleation events by which new steps are created. Randomly in time, $h(x, t)$ is changed to the new profile $\tilde{h}(x, t)$ such that for the increment $\delta h(x, t) = \tilde{h}(x, t) - h(x, t)$ one has $\delta h(x, t) = 1$ at some random point x', $|x'| \le t$, and $\delta h(x, t) = 0$ otherwise. Immediately after this nucleation event the deterministic evolution is followed until the next nucleation. Growing for a while the typical height profile has the shape of a droplet, $h(x, t) \simeq 2\sqrt{t^2 - x^2}$ for $|x| \le t$. Our interest are the statistical properties of the deviations from this average shape.

Warning: For the PNG model one has to specify the step speed and the intensity of the nucleation events. They can be adjusted to an arbitrary value through a linear scale change of space-time (x, t). Geometrically velocity one is distinguished and therefore adopted here. Intensity one for nucleation events seems also natural, but in fact introduces a string of factors $\sqrt{2}$. Therefore we deviate from previous conventions and set the intensity to be equal to 2.

The one-dimensional PNG model is just one model within the KPZ universality class for growth. However, for this model we have very refined statistical information, the most surprising breakthrough being the result of Baik, Deift, and Johansson [3], which states that

$$\lim_{t \to \infty} t^{-1/3}(h(0, t) - 2t) = \chi_2 \tag{1.1}$$

in distribution, cf. for the connection to the PNG model at the end of the introduction. χ_2 has the same distribution as the largest eigenvalue of a $N \times N$ random hermitian matrix (GUE) in the limit of $N \to \infty$. As discovered by Tracy and Widom [4] the distribution function $F_2(x) = \mathbb{P}(\chi_2 \le x)$ is governed by the Painlevé II equation. One has $F_2(x) = e^{-g(x)}$, where $g'' = u^2$, $g(x) \to 0$ as $x \to \infty$, and u is the global positive solution of $u'' = 2u^3 + xu$ (Painlevé II). Its asymptotics are $u(x) \simeq \sqrt{-x/2}$ for $x \to -\infty$, $u(x) \sim \mathrm{Ai}(x)$ for $x \to \infty$ with Ai the Airy function. In fact through a simple linear transformation [2] one obtains the height fluctuations

for any y, $|y| < 1$, as

$$\lim_{t\to\infty} t^{-1/3}\big(h(yt,t) - 2t\sqrt{1-y^2}\big) = (1-y^2)^{1/3}\chi_2\,. \tag{1.2}$$

At the next level of precision, one might wonder about joint distributions of $h(x,t)$ at the same time, say of $\{h(x_1,t), h(x_2,t)\}$. If $x_1 = y_1 t$ and $x_2 = y_2 t$, $y_1 \neq y_2$, then the heights are so far apart that they become statistically independent. Thus we have to consider closer by reference points. From the KPZ theory one knows that the lateral fluctuations live on the scale $t^{2/3}$. Therefore the natural object is

$$y \mapsto t^{-1/3}\big(h(yt^{2/3},t) - 2t\big) = h_t(y) \tag{1.3}$$

considered as a stochastic process in y. For large t we have $\langle h_t(y)\rangle \simeq -y^2$ and if in (1.2) we replace yt by $yt^{2/3}$ we obtain

$$\lim_{t\to\infty} h_t(y) = -y^2 + \chi_2 \tag{1.4}$$

in distribution, which suggests that $h_t(y) + y^2$ tends to a stationary stochastic process. As our main result we establish that this is indeed the case and rather explicitly identify the limit process. For reasons which will become clear in the sequel we call the novel limit process the Airy process and denote it by $A(y)$. We refer to Section 4 for its definition and to Section 5 for some of its properties. Somewhat compressed, one considers independent fermions in one space dimension as governed by the one-particle Hamiltonian

$$H = -\frac{d^2}{du^2} + u\,. \tag{1.5}$$

The fermions are in their ground state at zero chemical potential, which is the quasifree state determined by the spectral projection onto $\{H \leq 0\}$. Because of the linearly increasing potential there is a last fermion, which has the Tracy-Widom χ_2 as positional distribution. Extending to the Euclidean space-time through the propagator e^{-yH}, the fermions move along non-intersecting world lines with some suitable statistical weight. The Airy process $A(y)$ is the position of the last fermion at fermionic time y.

Theorem 1.1 *Let $A(y)$ be the stationary Airy process. Then in the sense of weak convergence of finite dimensional distributions*

$$\lim_{t\to\infty} h_t(y) = A(y) - y^2\,. \tag{1.6}$$

We recover the result in [3] as the special case of the convergence of the distributions for some fixed value of y. Even for fixed y we provide an alternative proof based on one-dimensional fermionic field theories.

As shown in [2, 5] the PNG droplet is isomorphic to a directed polymer with Poissonian point potential which in turn is related to the length of the longest increasing subsequence in a random permutation. For the sake of completeness we repeat our result in this language. We consider Poisson points of intensity 2 in the positive quadrant. A directed polymer is a piecewise linear path, ω, from $(0,0)$ to $(t+x, t-x)$, $|x| \leq t$, with each segment of ω bordered by Poisson points, under the constraint that their slope is in $[0,\infty]$. The length, $L(x,t,\omega)$, of the directed polymer ω is the number of Poisson points visited by ω. We set

$$L(x,t) = \max_{\omega} L(x,t,\omega)\,, \tag{1.7}$$

where the maximum is taken over all allowed directed paths at a fixed configuration of Poisson points and at specified endpoints. Then, in distribution,

$$L(x,t) = h(x,t)\,. \tag{1.8}$$

Therefore Theorem 1.1 yields the statistical properties of the length of the optimal path in dependence on its endpoint at transverse distance $yt^{2/3}$ away from (t,t).

To give a brief outline: In the following section we introduce the multi-layer PNG model, whose last layer is the PNG droplet. The multi-layer PNG has the remarkable property that the distribution at time t is the uniform distribution on all admissible height lines. Such kind of ensemble can be analyzed through Euclidean Fermi fields. Our case maps onto independent fermions on the lattice $\mathbb{Z}$ with the usual nearest neighbor hopping energy and subject to a linearly increasing external potential. The PNG droplet corresponds to the last fermionic world line. The convergence of the moments of the multi-layer PNG model in essence reduces to the convergence of the discrete Fermi propagator to the continuum Fermi propagator corresponding to the Hamiltonian (1.5). In the final section we establish some properties of the Airy process and discuss the two-point function $\langle (h(yt^{2/3},t) - h(0,t))^2 \rangle$.

2 The multi-layer PNG model

Inspired by the beautiful work of Johansson on the Aztec diamond [6] we enlarge the PNG droplet to the multi-layer PNG model. While this looks like a further complication, in fact the construction will provide us with a powerful machinery to analyze the statistics of the PNG droplet.

Instead of a single PNG line we now consider a collection of such lines, denoted by $h_\ell(x,t)$, $\ell \in \mathbb{N}_- = \{0,-1,-2,\dots\}$, taking values in $\mathbb{Z}$, which are arranged in ascending order, $h_\ell(x,t) < h_{\ell+1}(x,t)$, for all $x \in \mathbb{R}$, $t \geq 0$, cf. Figure 1 with a typical snapshot of the multi-layer PNG model at time t. The lines have jump size ± 1 and the total number of up- and down-steps is assumed to be finite. For each $t \geq 0$ one has $h_\ell(x,t) = \ell$ for $|x| \geq t$. To be definite we require $h_\ell(x,t) - \lim_{\epsilon \searrow 0} h_\ell(x \pm \epsilon, t) \in \{0,1\}$, which means that the height lines are upper semi-continuous. The set of all such height line configurations is denoted by Λ_t. The height lines

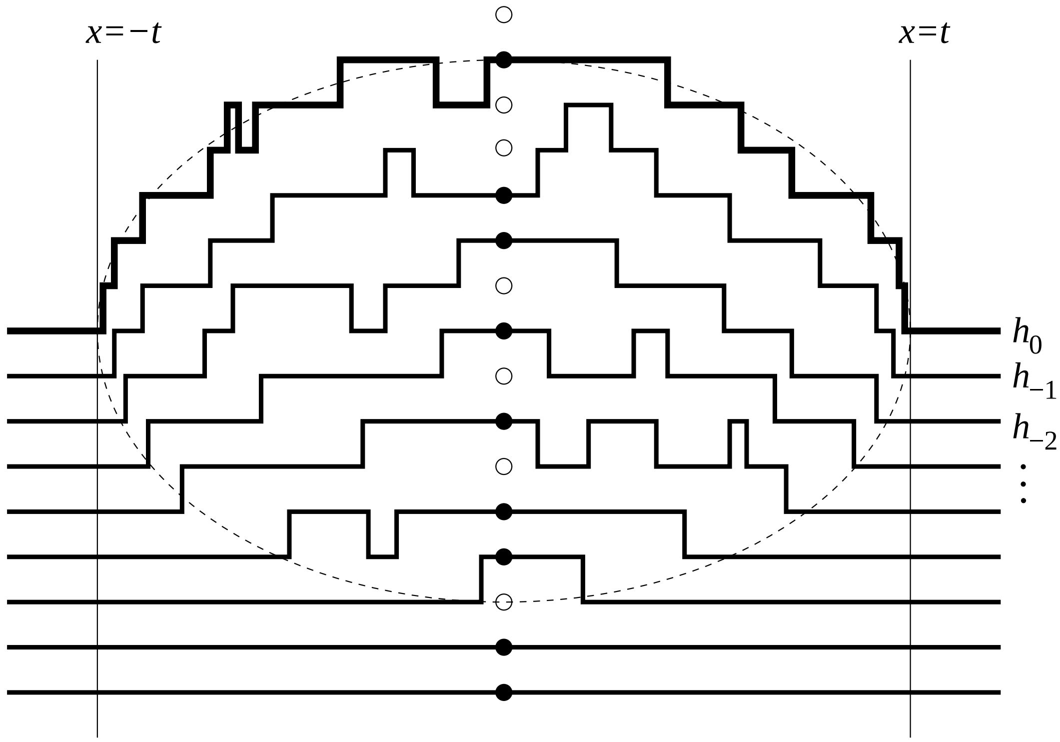

Figure 1: *A snapshot of a multi-layer PNG configuration at time t. As a guide to the eye the asymptotic droplet shape is indicated.*

evolve in time under a rule which is based on Viennot's geometric construction to prove the Robinson-Schensted(-Knuth) correspondence [9], hence called RSK dynamics by us. The initial line configuration is $h_\ell(x,0) = \ell$, $\ell \in \mathbb{N}_-$, for all x. Under the RSK dynamics only the region $|x| < t$ is modified. The top line evolves stochastically like the PNG droplet. In the lower lying lines the steps move and coalesce according to the PNG rules. However nucleations are determined by the annihilation events in the neighboring line above. Thus at time t if in the ℓ-th height line a collision of an up-step and a down-step occurs at position x, they disappear in this line only to reappear as nucleation at (x,t) in line $\ell - 1$. Clearly the RSK dynamics respects the ordering $h_\ell < h_{\ell+1}$.

To describe the configuration space of the multi-layer PNG model at time $t \geq 0$, we introduce the step positions as coordinates. Let n_ℓ be the number of up-steps in height line ℓ. Since $h_\ell(-t,t) = h_\ell(t,t)$, it must equal the number of down-steps in height line ℓ. If $n_\ell = 0$, $h_\ell(x,t) = \ell$ for all $x \in \mathbb{R}$. If $n_\ell > 0$, the position of the j-th up-step in height line ℓ is denoted by $y_j^{\ell,+}$, $-t < y_1^{\ell,+} < \cdots < y_{n_\ell}^{\ell,+} < t$ and the position of the j-th down-step in the same height line is denoted by $y_j^{\ell,-}$, $-t < y_1^{\ell,-} < \cdots < y_{n_\ell}^{\ell,-} < t$. We set $\mathbf{n} = (n_0, n_{-1}, \dots)$, and $|\mathbf{n}| = \sum_{\ell \leq 0} n_\ell$. For the RSK dynamics the total number of steps, $2|\mathbf{n}|$, is finite with probability one, since $|\mathbf{n}|$ equals the total number of nucleation events up to time t, i.e. the number of Poisson points in the triangle $\{(x',t') \big| \, |x'| < t',\, t' \leq t\}$. We denote by $\Gamma_t(\mathbf{n})$ the set of all step configurations $\big((y_j^{\ell,+}, y_j^{\ell,-})_{1 \leq j \leq n_\ell}\big)_{\ell \leq 0}$ resulting from an admissible line configuration $\big(h_\ell(x,t)\big)_{\ell \leq 0} \in \Lambda_t$. $\Gamma_t(0)$

is a single point and $\Gamma_t(\mathbf{n})$ is naturally embedded in $[-t,t]^{2|\mathbf{n}|}$. By definition of Λ_t, if an up-step and a down-step in the same height line are at the same location, they represent a nucleation and not a collision. Finally $\Gamma_t = \bigcup_{|\mathbf{n}|<\infty} \Gamma_t(\mathbf{n})$. We have thus defined the map $S : \Lambda_t \to \Gamma_t$, which we call step map. Clearly, S is invertible. The RSK dynamics on Λ_t induces a dynamics of steps on Γ_t. By construction the RSK dynamics stays inside Γ_t with probability one.

Remarkably, the distribution of the multi-layer PNG model at time t has a simple structure in being the uniform Lebesgue measure on all admissible step configurations.

Theorem 2.1 *Let w_t be the uniform measure on Γ_t, which means that $w_t\big(\Gamma_t(0)\big) = 1$ and $w_t \restriction \Gamma_t(\mathbf{n})$ is the $2|\mathbf{n}|$-dimensional Lebesgue measure on $\Gamma_t(\mathbf{n})$. Then $\int w_t = Z(t) = \exp(2t^2)$ and $\mu_t = Z(t)^{-1} w_t$ is a probability measure on Γ_t. If the height lines evolve under the RSK dynamics, then μ_t is the joint distribution of $\{h_\ell(x,t),\, x\in\mathbb{R},\, \ell\in\mathbb{N}_-\}$ under the step map S.*

Proof: The nucleation events determining the line configuration at time t are a Poisson process of intensity 2 in the triangle $\{(x',t')\big|\, |x'| < t',\, 0 \le t' \le t\}$. If (x',t') is a generic Poisson point, we label it through the new coordinates $(y^+,y^-) = \big(x' - (t-t'), x' + (t-t')\big)$. A Poisson point configuration consisting of N points is then given by $(y_i^+, y_i^-)_{i=1,\dots,N}$, such that $-t < y_1^+ < \cdots < y_N^+ < t$ and $y_i^+ < y_i^-$, $i = 1,\dots,N$. The set of all such point configurations is denoted by $\Delta_t(N)$, considered as a subset of $[-t,t]^{2N}$. We also set $\Delta_t = \bigcup_{N\ge 0} \Delta_t(N)$, with $\Delta_t(0)$ a single point. Δ_t inherits from the Poisson process the probability measure ν_t, where $\nu_t\big(\Delta_t(0)\big) = e^{-2t^2}$ and $\nu_t \restriction \Delta_t(N) = e^{-2t^2} dy_1^+ \cdots dy_N^+ dy_1^- \cdots dy_N^-$.

Next we define the growth map $G : \Delta_t \to \Lambda_t$. For given $(y_i^+, y_i^-)_{1\le i\le N} \in \Delta_t(N)$ we run the RSK dynamics to determine the line configuration at time t. Conversely for a given line configuration in Λ_t we run the RSK dynamics backwards in time which then determines the Poisson points corresponding to the nucleation events. Thus G is well defined. If all the line configurations with coinciding up- resp. down-step positions in different lines—a set of Lebesgue measure zero under the step map—are removed from Λ_t, then G is even bijective.

From the construction it is obvious that for $S\circ G : (y_i^+, y_i^-)_{i=1,\dots,N} \mapsto \big((y_j^{\ell,+}, y_j^{\ell,-})_{1\le j\le n_\ell}\big)_{\ell\le 0}$ with $N = |\mathbf{n}|$ one has the following set equalities, $\{y_i^+,\, i=1,\dots,N\} = \{y_j^{\ell,+},\, j=1,\dots,n_\ell,\, \ell\in\mathbb{N}_-\}$ and $\{y_i^-,\, i=1,\dots,N\} = \{y_j^{\ell,-},\, j=1,\dots,n_\ell,\, \ell\in\mathbb{N}_-\}$. Thus the map $S\circ G$ induces a mere relabeling of points. In particular ν_t is transformed to μ_t under $S\circ G$. $\Box$

Equipped with Theorem 2.1 the reader may jump ahead to Section 3 where the statistical properties of the line ensemble μ_t are studied. We take a little detour to report on two observations of interest. First there is a variant of the multi-layer PNG dynamics which was introduced by Gates and Westcott [7] in modeling crystal growth, hence called GW dynamics by us. Bulk properties of the GW dynamics are studied in [8] in the context of the anisotropic KPZ equation. Gates and Westcott regard the lines h_ℓ as contour lines of a crystal surface. The crystal is made up of atomic two-dimensional layers stacked along the z-axis. Layer 0 and below are completely filled. For layer 1 only the domain $\{(x,y)\in\mathbb{R}^2\big|\, y \le h_0(x,t)\}$ is filled with atoms, and in general, layer $-\ell+1$ is filled in the domain $\{(x,y)\in\mathbb{R}^2\big|\, y\le h_\ell(x,t)-\ell\}$, $\ell\in\mathbb{N}_-$.

The crystal is in contact with its supersaturated vapor. If overhangs are not permitted, it is a natural modeling assumption that each contour line grows under the PNG rules subject to a constraint of no touching. In this sense the GW dynamics is "more stochastic" than the RSK dynamics. Despite different rules, the RSK and GW dynamics yield an identical distribution for the line configurations at time t. Thus, alternatively, the proof of Theorem 2.1 could be based on the GW dynamics, cf. below.

A further observation is that the GW dynamics admits a discrete space-time version which, as to be discussed at the end of this section, inherits the simplicity of the distribution at time t. In fact, the discrete version of the GW dynamics is isomorphic to the shuffling algorithm for the Aztec diamond. This provides us with yet another approach to Theorem 2.1, namely to take the continuum limit of its discrete analogue.

Let us start with the GW dynamics. The top line h_0 evolves as the PNG droplet. With nucleation rate 2 pairs of up- and down-steps are generated in the forward light cone of the origin and move apart with velocity ± 1. Upon collision step pairs annihilate each other. The lower lines $h_{-1}, h_{-2}, \ldots$ follow the same dynamics, independently of each other, under the condition that nucleations are suppressed whenever they violate the monotonicity constraint, i.e. for height line ℓ, $\ell < 0$, nucleations occur only in the region $\{x \in \mathbb{R} \big| \, |x| < t, \, h_{\ell+1}(x,t) - h_\ell(x,t) \geq 2\}$ with space-time rate 2. As for the RSK dynamics, we have to convince ourselves that the GW dynamics lives on Γ_t with probability one. The simple procedure is to use duality. We regard the multi-layer PNG model as the stochastic evolution of the random field $\eta_t(j,x)$ over $\mathbb{Z} \times \mathbb{R}$ with values in $\{0,1\}$ by setting

$$\eta_t(j,x) = \begin{cases} 1, & \text{if } h_\ell(x,t) = j \text{ for some } \ell, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

The starting configuration is $\eta_0(j,x) = 1$ for $j \leq 0$, $\eta_0(j,x) = 0$ for $j \geq 1$. Let us introduce the flipped configuration through $\bar{\eta}_t(j,x) = 1 - \eta_t(j,x)$ with the corresponding height lines $\bar{h}_\ell(x,t), \ell = 1, 2, \ldots$. Then $\bar{h}_1(x,t)$ evolves again as the PNG droplet, now growing downwards towards negative j. Thus for the original process, at any given time t, there is a random index ℓ such that $h_\ell(x,t) = \ell$ for all x. This proves that at any time t with probability 1 there are only finitely many up- and down-steps.

Proposition 2.2 *Let $\{h_\ell(x,t), \, x \in \mathbb{R}, \, \ell \in \mathbb{N}_-\}$ be the height lines as generated by the GW dynamics. Under the step map S their joint distribution is μ_t, μ_t of Theorem 2.1.*

Proof: We did not discover a global version comparable to the proof of Theorem 2.1 and have to rely on an infinitesimal argument. Let $L(t)$ be the forward generator of the Markov jump process induced on Γ_t through the GW dynamics. We have to prove that

$$\frac{d}{dt}(e^{-2t^2} w_t) = L(t) e^{-2t^2} w_t \tag{2.2}$$

which means

$$-4tw_t = L(t)w_t\,, \tag{2.3}$$

since $dw_t/dt = 0$. $L(t)$ has four pieces. (i): There is a loss term $-4t$ due to nucleation events at $h_0(x,t)$ for $x \in [-t,t]$, which cancels the left hand side of (2.3). (ii): The free flow terms $\partial/\partial y_j^{\ell,+}$ of the up-step motion and $-\partial/\partial y_j^{\ell,-}$ of the down-step motion vanish when acting on the Lebesgue measure. There is no boundary contribution at $\pm t$, because the boundary moves with the same speed as the steps. (iii) and (iv): Let $r_\ell = |\{x \in \mathbb{R} \big|\, |h_\ell(x,t) - h_{\ell-1}(x,t)| \geq 2\}|$ with $|\cdot|$ denoting the one–dimensional Lebesgue measure and let $r = \sum_{\ell \leq 0} r_\ell$. For given $|\mathbf{n}|$ the current configuration gains in probability due to a transition from $|\mathbf{n}|+1$ to $|\mathbf{n}|$ through the collision of an up-step and a down-step. Their relative velocity is 2. Thus for a small time interval dt the gain is $2r\,dt$, since the ratio of the weight at $|\mathbf{n}|+1$ to the weight at $|\mathbf{n}|$ equals 1. There is a loss of probability due to nucleations in the current configuration. For the time dt it is $2r\,dt$, since nucleation events have intensity 2. Thus the gain term (iii) cancels exactly the loss term (iv). More extended versions of our argument can be found in [7, 8] where, however, a different geometry is discussed. □

Next we introduce a discrete time version of the multi-layer PNG. As explained in [6, 10] this model is isomorphic to the shuffling algorithm for the Aztec diamond. We discretize time, now denoted by $\tau = 0, 1, \ldots$. We also discretize the space axis in units of δ. As before, at time τ, the height lines are $h_\ell(x,\tau)$, $x \in \mathbb{R}$, $\ell \in \mathbb{N}$. $h_\ell(x,\tau) = \ell$ for $|x| \geq \delta\tau$. The non-crossing constraint $h_{\ell-1}(x,\tau) < h_\ell(x,\tau)$ is in force. Up- and down-steps are allowed only at midpoints of the form $(m+\frac{1}{2})\delta$, $m \in \mathbb{Z}$ and their distances must be odd, i.e. of the form $(2m+1)\delta$. To update from time τ to time $\tau+1$ only changes inside the strip $[-(\tau+1)\delta, (\tau+1)\delta]$ are allowed. The actual update consists of a deterministic and a stochastic step.
(i) *Deterministic step:* Given $h_\ell(x,\tau)$ every up-step is moved a δ-unit to the left, every down-step a δ-unit to the right. If at time τ there is a block of length 2δ, short a 2δ-block, with a down-step to the left of an up-step, then they annihilate each other, i.e. in this block $h_\ell(x,\tau)$ is replaced by its maximum. The configuration after the deterministic step is denoted by $\tilde{h}_\ell(x,\tau+1)$.
(ii) *Stochastic step:* The constant pieces of each height line $\tilde{h}_\ell(x,\tau+1)$ are subdivided in consecutive 2δ-blocks. To fix their location, the left endpoint of a 2δ-block either coincides with the right endpoint of a 2δ-block or is $\frac{1}{2}\delta$ away from an up-step, resp. from a down-step. If $\tilde{h}_0(x,\tau+1) = 0$, the 2δ-blocks are of the form $[(-\tau-1+2m)\delta, (-\tau-1+2m+2)\delta]$. If $\tilde{h}_\ell(x,\tau+1) = \ell$ for all x and $\tilde{h}_{\ell+1}(x,\tau+1) \neq \ell+1$ for some x, the 2δ-blocks of $\tilde{h}_\ell(x,\tau+1)$ start at $(y+\frac{1}{2})\delta$ with y the position of the first up-step (from the left) of $\tilde{h}_{\ell+1}(x,\tau+1)$. Finally we disregard those 2δ-blocks for which $\tilde{h}_{\ell+1}(x,\tau+1) - \tilde{h}_\ell(x,\tau+1) = 1$ for some x inside the block. After these preparations the stochastic update can be performed. Independently for each 2δ-block, we keep the original piece of the height line with probability $1-q$, $0<q<1$, and otherwise nucleate an up-step to the left and a down-step to the right midpoint of the two

adjacent δ-intervals. The line configuration after the stochastic update is denoted by $h_\ell(x,\tau{+}1)$.

In the limit of rare events the discrete multi-layer PNG model converges to its continuous time cousin. We set $t=\tau\delta$ and denote by $[t]$ the integer part of t. Then space-time is discretized in cells of lattice spacing δ. A nucleation event covers a block of two adjacent cells. If we set $q=4\delta^2$, then in the limit $\delta\to 0$ we obtain a Poisson process of intensity 2. Therefore in this limit $h_\ell(x,[t/\delta])\to h_\ell(x,t)$ as a stochastic process.

The discretized multi-layer PNG model inherits the simplicity of the time τ measure, $\mu(\tau)$. The height line $h_\ell(x,\tau)$ has n_ℓ up-steps. The total number of up-steps is then $\sum_{\ell\leq 0} n_\ell = n$. To the collection of height lines $\{h_\ell(x,\tau),\, x\in\mathbb{R},\, \ell\leq 0\}$ we assign the weight $(q/(1-q))^n$. The partition function, $Z_{\rm d}(\tau)$, is the sum over all weights. We set $Z_{\rm d}(\tau)^{-1}=\mathbb{P}\big(\{h_0(x,0)=0\}\big)$. Therefore

$$Z_{\rm d}(\tau)=\prod_{j=1}^{\tau}(1-q)^{-j}=(1-q)^{-\tau(\tau+1)/2}\,. \tag{2.4}$$

If the weight at time τ is denoted by $w(\tau)$, we claim that

$$\mu(\tau)=Z_{\rm d}(\tau)^{-1}w(\tau) \tag{2.5}$$

is the time τ probability measure of the discrete multi-layer PNG. Let K_τ be the transition kernel from τ to $\tau{+}1$, as explained in steps (i), (ii) above. We have to show $\mu(\tau{+}1)=\mu(\tau)K_\tau$, equivalently

$$(1-q)^{\tau+1}w(\tau+1)=w(\tau)K_\tau\,. \tag{2.6}$$

(2.6) is established in Proposition 2.3 below. But first we want to convince ourselves that $\mu(\tau)$ yields μ_t of the continuous time PNG in the limit $\delta\to 0$. We note that for $\tau=[t/\delta], q=4\delta^2$,

$$\lim_{\delta\to 0} Z_{\rm d}([t/\delta])=\lim_{\delta\to 0}\exp\big(-\tfrac{1}{2}[t/\delta]([t/\delta]+1)\log(1-4\delta^2)\big)=e^{2t^2}=Z(t)\,. \tag{2.7}$$

A configuration with n up/down-step pairs has the weight $(q/1-q)^n\cong(4\delta^2)^n$. Because of the constraint in the up-step locations, in the limit $\delta\to 0$ the weight converges to the $2n$-dimensional Lebesgue measure constrained to $\Gamma_t(\mathbf{n})$, $n=|\mathbf{n}|$. Thus $\mu([t/\delta])\to\mu_t$ as $\delta\to 0$, as it should be.

Proposition 2.3 *Let the weight, $w(\tau)$, of the height lines of the discrete multi-layer PNG be given by $(q/(1-q))^n$, where n is the total number of up-steps (equivalently down-steps). Then (2.6) holds.*

Proof: Let $w(\tau+1)$ be the weight for the configuration $h_\ell(x,\tau+1)$. We construct from it the configuration $\tilde{h}_\ell(x,\tau{+}1)$ by removing all spikes from $h_\ell(x,\tau{+}1)$, i.e. all 2δ-blocks containing an up-step to the left and a down-step to the right. Let s_ℓ be the number of spikes for $h_{\ell+1}(x,\tau{+}1)$,

let n_ℓ be the number of up-steps for $\tilde{h}_\ell(x, \tau+1)$, and let b_ℓ be the number of 2δ-blocks with $\tilde{h}_\ell(x,\tau+1)$ constant such that $\tilde{h}_{\ell+1}(x,\tau+1) - \tilde{h}_\ell(x,\tau+1) \geq 2$ within that block, b_0 is the number of flat 2δ-blocks of $\tilde{h}_0(x,\tau+1)$. Next we map the configuration $\tilde{h}_\ell(x,\tau+1)$ to the configuration $\tilde{h}_\ell(x,\tau)$ by moving all up-down-steps one step backwards in time. By construction, n_ℓ does not change. Let a_ℓ be the number of downwards open blocks, i.e. flat 2δ-blocks of $\tilde{h}_\ell(x,\tau)$ such that $h_{\ell-1}(x,\tau)$ has distance ≥ 2 within that block.

The transition kernel $K_\tau = K_\mathrm{d} K_\mathrm{s}$, where K_d is the deterministic step (i) and K_s is the stochastic step (ii). To compute $w(\tau)K_\tau$, we first evaluate $w(\tau)K_\mathrm{d}$ in the configuration $\tilde{h}_\ell(x,\tau+1)$. We have to sum over all line configurations $h_\ell(x,\tau)$ leading to $\tilde{h}_\ell(x,\tau+1)$ under K_d. A downwards open block of $\tilde{h}_\ell(x,\tau)$ had either no steps, weight 1, or a downwards spike, weight $q/(1-q)$. Summing over these 2^{a_ℓ} possibilities results in the weight $(1-q)^{-a_\ell}\big(q/(1-q)\big)^{n_\ell}$ for $\tilde{h}_\ell(x,\tau+1)$. Applying the stochastic transition K_s yields the weight $w(\tau)K_\tau$ evaluated at $h_\ell(x,\tau+1)$ as

$$q^{s_\ell}(1-q)^{b_\ell - s_\ell}(1-q)^{-a_\ell}(q/(1-q))^{n_\ell}\,. \tag{2.8}$$

On the other hand, according to $w(\tau+1)$, $h_\ell(x,\tau+1)$ has the weight $\big(q/(1-q)\big)^{s_\ell+n_\ell}$. Comparing with (2.6) and (2.8), we have to prove

$$\sum_{\ell\leq 0}(b_\ell - a_\ell) = \tau + 1\,. \tag{2.9}$$

Let N be the index of the last height line for which $h_N(x,\tau) = N$ for all x. Then $a_N = 0$. For two adjacent lines it is easily verified that

$$b_\ell - a_{\ell+1} = n_{\ell+1} - n_\ell, \quad \ell \leq -1\,. \tag{2.10}$$

Inserting in the left side of (2.9) and using $n_N = 0$, the telescoping sum gives $b_0 + n_0$, which by definition is independent of $\tilde{h}_0(x,\tau+1)$ and equals $\tau+1$. $\Box$

3 $1+1$-dimensional Fermi field

w_t is the uniform distribution on all allowed line configurations of the continuous time multi-layer PNG. Except for exclusion (entropic repulsion), the height lines do not interact. Such a statistical mechanics system is most conveniently analyzed through the transfer matrix method. Its implementation requires the height lines to be restricted to a bounded interval $\{-M, -M+1, \dots, M\} = I_M$. The case of interest is then obtained in the limit as $M \to \infty$. To explain the principle, we omit the argument t and label the height lines more conventionally as $h_\ell(x)$, $\ell = 1, \dots, N$, $N \leq 2M+1$, with $x \in [0,t]$. The height lines are constrained through $-M \leq h_1(x) < \cdots < h_N(x) \leq M$ for all $x \in [0,t]$. In addition we fix the initial configuration q and the final configuration q', i.e. $h_\ell(0) = q_\ell$, $h_\ell(t) = q'_\ell$, $\ell = 1, \dots, N$. As before under the step map the ℓ-th height line is specified by the location of the up-steps, $0 < y_1^{\ell,+} < \cdots < y_{n_\ell}^{\ell,+} < t$, and

the location of the down-steps, $0 < y_1^{\ell,-} < \cdots < y_{n'_\ell}^{\ell,-} < t$, where $n_\ell \neq n'_\ell$ is allowed. Admissible line configurations are assumed to have a uniform weight, which means that a small volume element has the weight $\prod_{\ell=1}^{N}\prod_{j=1}^{n_\ell}\prod_{j'=1}^{n'_\ell} dy_j^{\ell,+}dy_{j'}^{\ell,-}$. The configuration with no steps has weight 1.

We want to compute the partition function $Z_t(q,q')$ which is defined as the weight integrated over all admissible step configurations. For this purpose the simplex $\Omega_N = \{q \in \mathbb{Z}^N \big| -M \leq q_1 < \cdots < q_N \leq M\}$ is introduced. Clearly $q, q' \in \Omega_N$ and we regard $Z_t(q,q')$ as a $|\Omega_N| \times |\Omega_N|$ matrix. By the product property of the Lebesgue measure Z_t satisfies the semigroup property

$$Z_t Z_s = Z_{t+s}, \quad t,s \geq 0,\ Z_0 = \mathbb{1}, \tag{3.1}$$

$\mathbb{1}$ the identity matrix. Thus there exists an infinitesimal generator G_N, such that

$$Z_t = e^{-tG_N}, \quad t \geq 0. \tag{3.2}$$

Differentiating at $t = 0$ one concludes that G_N acting on functions f on Ω_N is given by

$$G_N f(q) = -\sum_{q' \in \Omega_N} c(q,q') f(q'), \tag{3.3}$$

where for $q, q' \in \Omega_N$

$$c(q,q') = \begin{cases} 1 & \text{if } \sum_{\ell=1}^{N} |q_\ell - q'_\ell| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Computationally much more powerful is to impose the constraint of no overlap through antisymmetry. Let $\mathcal{F}_N$ be the subspace of $\ell_2\big((I_M)^N\big)$ consisting of antisymmetric functions over $(I_M)^N$, i.e. $f \in \mathcal{F}_N$ satisfies

$$f(q_1,\ldots,q_N) = (-1)^{\operatorname{sign}\pi} f(q_{\pi(1)},\ldots,q_{\pi(N)}) \tag{3.5}$$

for every permutation π. $\mathcal{F}_N$ is equipped with the canonical basis $f_q, q \in \Omega_N$, defined through

$$f_q(q') = \frac{1}{\sqrt{N!}} \sum_{\pi} (-1)^{\operatorname{sign}\pi} \delta_q(q'_{\pi(1)},\ldots,q'_{\pi(N)}) \tag{3.6}$$

with $\delta_q(q') = 1$ if $q = q'$ and $\delta_q(q') = 0$ otherwise. The normalization is chosen such that $\langle f_q, f_{q'}\rangle = \delta_q(q')$, with $\langle\cdot,\cdot\rangle$ denoting the scalar product in $\mathcal{F}_N$. Let us also define the one-particle Hamiltonian

$$\begin{aligned} H_{\mathrm{d}}^M \psi(-M) = -\psi(-M+1), \quad H_{\mathrm{d}}^M \psi(M) = -\psi(M-1), \\ H_{\mathrm{d}}^M \psi(n) = -\psi(n+1) - \psi(n-1) \quad \text{for } |n| < M, \end{aligned} \tag{3.7}$$

as acting on functions ψ over I_M. The corresponding N-particle Hamiltonian, is then given through

$$H_{\mathrm{d},N}^M = \sum_{j=1}^{N} 1 \otimes \cdots \otimes H_{\mathrm{d}}^M \otimes \cdots \otimes 1, \tag{3.8}$$

where H_{d}^M is inserted at the j-th position of the N-fold product. Clearly for $f \in \mathcal{F}_N$ one has $H_{\mathrm{d},N}^M f \in \mathcal{F}_N$ and $H_{\mathrm{d},N}^M$ is regarded as acting on $\mathcal{F}_N$. With these notations one has the identity

$$\langle f_q, e^{-tH_{\mathrm{d},N}^M} f_{q'} \rangle = e^{-tG_N}(q, q') = Z_t(q, q') \tag{3.9}$$

for $q, q' \in \Omega_N$.

At this point it is more convenient to switch to fermionic language which is devised precisely to take the antisymmetry into account. The CAR algebra over I_M is generated by $a^*(j)$, $a(j)$, $j \in I_M$. They satisfy the canonical anticommutation relations

$$\{a(i), a^*(j)\} = \delta_{ij}, \quad \{a(i), a(j)\} = 0, \quad \{a^*(i), a^*(j)\} = 0, \tag{3.10}$$

$i, j = -M, \dots, M$, $\{A, B\} = AB + BA$. In the Fock representation the algebra is realized as operators on the antisymmetric Fock space $\mathcal{F}$ over I_M,

$$\mathcal{F} = \bigoplus_{N=0}^{2M+1} \mathcal{F}_N. \tag{3.11}$$

The second quantization of the $(2M+1) \times (2M+1)$ matrix H_{d}^M is defined by

$$\widehat{H}_{\mathrm{d}}^M = \sum_{i,j \in I_M} a^*(i) \big(H_{\mathrm{d}}^M\big)_{ij} a(j). \tag{3.12}$$

$\widehat{H}_{\mathrm{d}}^M$ restricted to $\mathcal{F}_N$ agrees with $H_{\mathrm{d},N}^M$. From (3.9) one concludes that as a fermionic operator the transfer matrix is given through

$$e^{-t\widehat{H}_{\mathrm{d}}^M}, \quad t \ge 0, \tag{3.13}$$

which covers all $0 \le N \le 2M+1$.

We exploit the new flexibility by assuming that $q, q' \in \Omega = \bigcup_{N=0}^{2M+1} \Omega_N$, which is identified with $\{0,1\}^{I_M}$. The case of interest is $q = q'$ and each boundary configuration q has the weight $\prod_{j=-M}^{M} \exp\big(\lambda(j) \sum_{i=1}^{N} \delta_{q_i,j}\big)$. In other words the configuration q has a product weight with factor $e^{\lambda(j)}$ if site j is occupied and factor 1 if the site j is empty. The corresponding partition function is then given through

$$Z_M^\lambda = \mathrm{tr}\big[e^{\widehat{N}^M} e^{-t\widehat{H}_{\mathrm{d}}^M}\big], \tag{3.14}$$

where the trace is over $\mathcal{F}$ and

$$\widehat{N}^M = \sum_{j \in I_M} \lambda(j) a^*(j) a(j). \tag{3.15}$$

The probability that site j at x is occupied, i.e. $h_\ell(x) = j$ for some ℓ, is obtained from the transfer matrix as

$$\big(Z_M^\lambda\big)^{-1} \mathrm{tr}\big[e^{\widehat{N}^M} e^{-x\widehat{H}_{\mathrm{d}}^M} a^*(j) a(j) e^{-(t-x)\widehat{H}_{\mathrm{d}}^M}\big] = \mathrm{tr}[\widehat{\rho}_x^M a^*(j) a(j)], \tag{3.16}$$

with the density matrix

$$\widehat{\rho}_x^M = \left(Z_M^\lambda\right)^{-1} e^{-(t-x)\widehat{H}_{\mathrm{d}}^M} e^{\widehat{N}^M} e^{-x\widehat{H}_{\mathrm{d}}^M}, \tag{3.17}$$

$0 \leq x \leq t$.

Exponentials of operators quadratic in a, a^* are easily handled. If A is a $(2M+1)\times(2M+1)$ matrix with second quantization

$$\widehat{A} = \sum_{i,j\in I_M} a^*(i) A_{ij} a(j), \tag{3.18}$$

then

$$e^{\widehat{A}} = e^A \otimes \cdots \otimes e^A \tag{3.19}$$

on $\mathcal{F}_N$. This implies

$$\mathrm{tr}[e^{\widehat{A}}] = \det(1 + e^A) = Z_A, \tag{3.20}$$

compare with [11]. Let us set $\widehat{\rho}_A = (Z_A)^{-1} e^{\widehat{A}}$ as density matrix. The two-point function has the form

$$R(i,j) = \mathrm{tr}[\widehat{\rho}_A a^*(i) a(j)] = \Big((1 + e^{-A})^{-1}\Big)_{ij} \tag{3.21}$$

and more generally

$$\mathrm{tr}[\widehat{\rho}_A a^*(i_1)\cdots a^*(i_m) a(j_n)\cdots a(j_1)] = \delta_{m,n} \det\big(R(i_k, j_{k'})\big)_{k,k'=1,\ldots,m}. \tag{3.22}$$

The expectations of other monomials are determined by means of the anticommutation relations (3.10). One may regard $\mathrm{tr}[\widehat{\rho}_A \,\cdot\,] = \omega_A(\,\cdot\,)$ as a linear functional on the CAR algebra. By definition $\omega_A(1) = 1$. If in addition ω_A is positive, then ω_A is called a quasifree state [15].

As can be seen from (3.19) products of exponentials follow the same pattern. For the $(2M+1)\times(2M+1)$ matrices A, B we set

$$e^A e^B = e^C. \tag{3.23}$$

Then

$$e^{\widehat{A}} e^{\widehat{B}} = e^{\widehat{C}} \tag{3.24}$$

with $\widehat{\cdot}$ defined as in (3.18).

Each $a^*(i)a(i)$ is a symmetric projection and thus has eigenvalues in $\{0,1\}$. $\{a^*(i)a(i),\, i \in I_M\}$ is a family of commuting operators. Under $\widehat{\rho}_x^M$ they have a joint signed spectral measure which by construction is a probability measure on $\{0,1\}^{I_M}$. By (3.22) it is of determinantal form, compare also with (3.54) below. Thus under $\widehat{\rho}_x^M$ the family $\{a^*(i)a(i),\, i \in I_M\}$ is a point

process on I_M. In the probabilistic literature point processes of this structure are known as determinantal [12].

Our construction extends to occupation variables depending on several x. For example, the probability that site j at x and site i at y, $0 < x < y < t$, are both occupied is obtained through the transfer matrix as

$$\big(Z_M^\lambda\big)^{-1}\mathrm{tr}\big[e^{\widehat{N}^M} e^{-x\widehat{H}_{\mathrm{d}}^M} a^*(j)a(j)e^{-(y-x)\widehat{H}_{\mathrm{d}}^M} a^*(i)a(i)e^{-(t-y)\widehat{H}_{\mathrm{d}}^M}\big]. \tag{3.25}$$

Such expressions can be computed using (3.24) and (3.23), and the rules

$$e^{t\widehat{A}}a(j)e^{-t\widehat{A}} = \sum_{i\in I_M}(e^{-tA})_{ji}a(i), \quad e^{t\widehat{A}}a^*(j)e^{-t\widehat{A}} = \sum_{i\in I_M} a^*(i)(e^{tA})_{ij}. \tag{3.26}$$

With these preparations we return to the PNG droplet. Recall the definition (2.1) of the random field of occupation variables $\eta_t(j,x)$, $j\in\mathbb{Z}$, $x\in\mathbb{R}$: $\eta_t(j,x)=1$ if $h_\ell(x,t)=j$ for some ℓ and $\eta_t(j,x)=0$ otherwise. By definition $\eta_t(j,x)=0$ for $j\geq 1$, $|x|\geq t$, and $\eta_t(j,x)=1$ for $j\leq 0$, $|x|\geq t$. The joint distribution of $\eta_t(j,x)$, $j\in\mathbb{Z}$, $x\in\mathbb{R}$, is induced through the probability measure μ_t with expectation denoted by $\mathbb{E}_t$. Our first goal is to obtain the joint distribution of $\eta_t(j,0)$, $j\in\mathbb{Z}$. From the considerations above, it is obvious that this point process is determinantal. The only remaining task is to compute the two-point function and to study its limit behavior. One should pay attention to a minor linguistic problem. The meaning of the time parameter t of the PNG model is now reduced to a mere scaling parameter. j labels fermionic space and x stands for fermionic time. Thus space-time means now $\mathbb{Z}\times\mathbb{R}$.

Let us start by explaining the result for moments at $x=0$. We introduce the limit $M\to\infty$ of the one-particle Hamiltonian in (3.7) as

$$H_{\mathrm{d}}\psi(n) = -\psi(n+1)-\psi(n-1). \tag{3.27}$$

In addition, we add a linear potential of slope $1/t$ to define

$$H_t\psi(n) = -\psi(n+1)-\psi(n-1)+\frac{n}{t}\psi(n), \tag{3.28}$$

regarded as an operator on $\ell_2=\ell_2(\mathbb{Z})$. H_t has the complete set of eigenfunctions $\varphi^{(l)}(n) = J_{n-l}(2t)$, $l\in\mathbb{Z}$, with eigenvalues $\varepsilon_l=\frac{l}{t}$, $H_t\varphi^{(l)}=\varepsilon_l\varphi^{(l)}$, $\langle\varphi^{(l)},\varphi^{(l')}\rangle=\delta_{l\,l'}$, $\langle\cdot,\cdot\rangle$ denoting the scalar product. Here $J_n(z)$ is the Bessel function of integer order n and we follow throughout the conventions of [14], Chapter 9. We will need the spectral projection B_t onto $\{H_t\leq 0\}$. In position space its integral kernel is the discrete Bessel kernel

$$B_t(i,j) = \sum_{l\leq 0} J_{i-l}(2t)J_{j-l}(2t). \tag{3.29}$$

Using that $H_t\,\varphi^{(l)}=\varepsilon_l\,\varphi^{(l)}$, (3.29) can be converted into a telescoping sum with the result

$$B_t(i,j) = \frac{t}{i-j}\big(J_{i-1}(2t)J_j(2t)-J_i(2t)J_{j-1}(2t)\big) \tag{3.30}$$

for $i \neq j$ and on the diagonal

$$B_t(i,i) = t\big(L_{i-1}(2t)J_i(2t) - L_i(2t)J_{i-1}(2t)\big), \tag{3.31}$$

where $L_j(2t) = \frac{d}{dj} J_j(2t)$. We also introduce the CAR algebra $\mathcal{A}_{\mathrm{d}}$ over $\mathbb{Z}$. It is generated by the operators $a(j), a^*(j)$, $j \in \mathbb{Z}$, satisfying the canonical anticommutation relations (3.10). Let ω_t be the quasifree state on the CAR algebra $\mathcal{A}_{\mathrm{d}}$ defined through $\omega_t\big(a(j)\big) = 0 = \omega_t\big(a^*(j)\big)$ and the two-point function

$$\omega_t\big(a^*(i)a(j)\big) = B_t(i,j), \tag{3.32}$$

which means that higher order monomials satisfy (3.22) with R replaced by B_t [15]. ω_t is the ground state for non-interacting fermions with one-particle Hamiltonian (3.28) at zero chemical potential.

Theorem 3.1 *We have*

$$\mathbb{E}_t\Big(\prod_{k=1}^m \eta_t(j_k,0)\Big) = \omega_t\Big(\prod_{k=1}^m a^*(j_k)a(j_k)\Big) = \det\big(B_t(j_k,j_{k'})\big)_{1\le k,k'\le m}\,, \tag{3.33}$$

the second equality being valid only for pairwise distinct points $j_1,\ldots,j_m$.

Proof: The state ω_t has an infinite number of fermions and cannot be represented as a vector in Fock space. Thus we first have to constrain to finite volume I_M, such that $|h_\ell(x,t)| \le M, \ell = 0,\ldots,-M$. The corresponding uniform distribution is denoted by μ_t^M. Clearly μ_t^M converges to μ_t as $M \to \infty$. By construction $h_\ell(t,t) = h_\ell(-t,t) = \ell$, $\ell = 0,-1,-2,\ldots$. In other words $\eta_t(j,-t) = \eta_t(j,t) = 1$ for $j \le 0$ and $= 0$ for $j \ge 1$. It is convenient to approximate these boundary configurations through the weight e^β, $-M \le j \le 0$, weight $e^{-\beta}$, $1 \le j \le M$, for an occupied site and weight 1 for an empty site in the limit $\beta \to \infty$. Thus, if we set $\widehat{N}^M$ as in (3.15) with $\lambda(j) = 1$ for $-M \le j \le 0$ and $\lambda(j) = -1$ for $1 \le j \le M$, we have

$$\mathbb{E}_t^M\Big(\prod_{k=1}^m \eta_t(j_k,0)\Big) = \lim_{\beta\to\infty} \frac{1}{Z(\beta)} \mathrm{tr}[e^{\beta\widehat{N}^M} e^{-t\widehat{H}_{\mathrm{d}}^M} \prod_{k=1}^m a^*(j_k)a(j_k) e^{-t\widehat{H}_{\mathrm{d}}^M}] \tag{3.34}$$

with $Z(\beta)$ the normalizing partition function. Since the moments of a quasifree state are determined by the two-point function, to prove (3.33) it suffices to consider the expectation of $a^*(j)a(i)$ and to subsequently take the limit $M \to \infty$. Let P_-^M be the projection onto $\{-M,\ldots,0\}$, P_+^M onto $\{1,\ldots,M\}$, $P_+^M + P_-^M$ being the identity. Then

$$\begin{aligned}
&\lim_{\beta\to\infty} Z(\beta)^{-1}\mathrm{tr}\big[e^{\beta\widehat{N}^M} e^{-t\widehat{H}_{\mathrm{d}}^M} a^*(j)a(i) e^{-t\widehat{H}_{\mathrm{d}}^M}\big] \\
&\quad = \lim_{\beta\to\infty} \Big(\big(1 + e^{tH_{\mathrm{d}}^M}(e^\beta P_+^M + e^{-\beta}P_-^M)e^{tH_{\mathrm{d}}^M}\big)^{-1}\Big)_{ij} \\
&\quad = \Big(e^{-tH_{\mathrm{d}}^M} P_-^M (P_+^M + P_-^M e^{-2tH_{\mathrm{d}}^M} P_-^M)^{-1} P_-^M e^{-tH_{\mathrm{d}}^M}\Big)_{ij}\,.
\end{aligned} \tag{3.35}$$

If P_- denotes the projection onto $\mathbb{N}_-$, $P_+ = 1 - P_-$, then

$$\lim_{M\to\infty} P_\pm^M = P_\pm, \qquad \lim_{M\to\infty} e^{tH_\mathrm{d}^M} = e^{tH_\mathrm{d}}. \tag{3.36}$$

To prove the theorem we only have to check the identity

$$e^{-tH_\mathrm{d}} P_-(P_+ + P_- e^{-2tH_\mathrm{d}} P_-)^{-1} P_- e^{-tH_\mathrm{d}} = B_t \tag{3.37}$$

as an operator identity on ℓ_2.

We define the left shift D, $D\psi(n) = \psi(n+1)$, and the adjoint right shift D^*, $D^*\psi(n) = \psi(n-1)$. Clearly $[D, D^*] = 0$. One has $H_\mathrm{d} = -D - D^*$. Using $2\frac{d}{dt}J_n(t) = J_{n-1}(t) - J_{n+1}(t)$, one obtains

$$\frac{d}{dt} B_t = (D^* - D)B_t - B_t(D^* - D)\,. \tag{3.38}$$

Integrating with the initial condition $B_0 = P_-$ yields

$$B_t = e^{t(D^*-D)} P_- e^{-t(D^*-D)} \tag{3.39}$$

and

$$e^{tH_\mathrm{d}} B_t e^{tH_\mathrm{d}} = e^{-t(D+D^*)} e^{t(D^*-D)} P_- e^{-t(D^*-D)} e^{-t(D+D^*)} = e^{-2tD} P_- e^{-2tD^*}\,. \tag{3.40}$$

Therefore (3.37) is equivalent to

$$e^{-2tD} P_- e^{-2tD^*} = (P_- e^{2tD^*} e^{2tD} P_-)^{-1} \tag{3.41}$$

as an operator identity on $P_-\ell_2$. We decompose our space as $\ell_2 = P_+\ell_2 \oplus P_-\ell_2$. Then with the definition

$$e^{-2tD} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \tag{3.42}$$

we have

$$e^{-2tD} P_- e^{-2tD^*} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ 0 & c^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & c^*c \end{pmatrix}. \tag{3.43}$$

Using the splitting of (3.42), one constructs the inverse operators e^{2tD}, e^{2tD^*}. By a straightforward computation one obtains

$$P_- e^{2tD^*} e^{2tD} P_- = \begin{pmatrix} 0 & 0 \\ 0 & (c^*c)^{-1} \end{pmatrix}. \quad \square \tag{3.44}$$

From Theorem 3.1 one immediately infers the distribution of the height of the PNG droplet at $x = 0$. Clearly

$$\begin{aligned} \mathbb{P}_t\big(\{h(0,t) < n\}\big) &= \mathbb{P}_t\big(\{\eta_t(j,0) = 0 \text{ for all } j \geq n\}\big) \\ &= \lim_{\beta\to\infty} \omega_t\Big(\prod_{j=n}^{\infty} e^{-\beta a^*(j)a(j)}\Big) = \lim_{\beta\to\infty} \det\big(1 - (1-e^{-\beta})P_n B_t\big) \\ &= \det(1 - P_n B_t), \end{aligned} \tag{3.45}$$

where P_n denotes the projection onto $\{n, n+1, \dots\}$ in ℓ_2. Since $L(0,t) = h(0,t)$, see eq. (1.8), we have rederived that the length of the longest increasing subsequence of a Poissonized random permutation has a distribution linked to the discrete Bessel kernel. Previous proofs take the route via the Plancherel measure. We refer to [16]. It would be of interest to better understand how these proofs are linked to the multi-layer PNG.

So far we considered only the distribution of $\eta_t(j,x)$ at $x=0$. The transfer matrix method can handle also the distribution referring to several x, like the joint distribution of $\eta_t(i,0)$, $\eta_t(j,x)$, see eq. (3.25). The transfer matrix is generated by the Hamiltonian H_d of (3.27). As in the case of fixed x, the joint moments have determinantal form with the entries given by the Euclidean Fermi propagator $B_t(j,x;j',x')$. Following the scheme in (3.34) it is defined through a finite volume approximation, $B_t(j,x;j',x') = \lim_{M\to\infty}\lim_{\beta\to\infty} B_t^{M\beta}(j,x;j',x')$, where

$$B_t^{M\beta}(j,x;j',x') = \begin{cases} Z(\beta)^{-1}\mathrm{tr}\big[e^{\beta\widehat{N}^M} e^{-t\widehat{H}_\mathrm{d}^M}\big(e^{-x\widehat{H}_\mathrm{d}^M}a^*(j)e^{x\widehat{H}_\mathrm{d}^M}\big)\big(e^{-x'\widehat{H}_\mathrm{d}^M}a(j')e^{x'\widehat{H}_\mathrm{d}^M}\big)e^{-t\widehat{H}_\mathrm{d}^M}\big] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } -t\le x\le x'\le t, \\ -Z(\beta)^{-1}\mathrm{tr}\big[e^{\beta\widehat{N}^M} e^{-t\widehat{H}_\mathrm{d}^M}\big(e^{-x'\widehat{H}_\mathrm{d}^M}a(j')e^{x'\widehat{H}_\mathrm{d}^M}\big)\big(e^{-x\widehat{H}_\mathrm{d}^M}a^*(j)e^{x\widehat{H}_\mathrm{d}^M}\big)e^{-t\widehat{H}_\mathrm{d}^M}\big] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } -t\le x'< x\le t. \end{cases} \tag{3.46}$$

Note that time order must be respected in such a way that there are only decaying exponentials. The minus sign in (3.46) for $x'<x$ results from commuting a^* and a. At coinciding arguments the definition conforms with $\mathbb{E}\big(\eta_t(j,x)\big) = B_t(j,x;j,x)$. Using (3.26) one obtains

$$B_t(j,x;j',x') = \big(e^{-xH_\mathrm{d}}(B_t - \mathbb{1}\,\theta(x-x'))e^{x'H_\mathrm{d}}\big)_{jj'}, \tag{3.47}$$

for $|x|\le t$, $|x'|\le t$, $x\neq x'$, with the step function $\theta(x)=0$ for $x<0$, $\theta(x)=1$ for $x>0$. B_t has a jump discontinuity at $x=x'$. For coinciding time arguments one has

$$B_t(j,x;j',x) = (e^{-xH_\mathrm{d}}B_t e^{xH_\mathrm{d}})_{jj'}. \tag{3.48}$$

For later use the propagator is rewritten in the eigenbasis of H_t. The integer order Bessel function has the representation

$$J_n(2t) = \frac{1}{2\pi i}\oint \frac{dz}{z} e^{t(z^{-1}-z)} z^n \tag{3.49}$$

where the contour integration is a circle around $z=0$. Therefore

$$\big(e^{-xH_\mathrm{d}}J_\cdot(2t)\big)_n = \frac{1}{2\pi i}\oint \frac{dz}{z} e^{t(z^{-1}-z)}e^{x(z^{-1}+z)}z^n\,. \tag{3.50}$$

Substituting z by $(t+x)^{1/2}(t-x)^{1/2}z$ yields

$$\big(e^{-xH_\mathrm{d}}J_\cdot(2t)\big)_n = J_n\big(2\sqrt{t^2-x^2}\big)\Big(\frac{t+x}{t-x}\Big)^{n/2} \tag{3.51}$$

and thus for $x \neq x'$

$$B_t(j,x;j',x') = \sum_{l\in\mathbb{Z}} \mathrm{sgn}(x'-x)\theta\big((x-x')(l+\tfrac{1}{2})\big)\Big(\frac{t+x}{t-x}\Big)^{(j-l)/2} J_{j-l}\big(2\sqrt{t^2-x^2}\big) \times J_{j'-l}\big(2\sqrt{t^2-x'^2}\big)\Big(\frac{t-x'}{t+x'}\Big)^{(j'-l)/2}. \tag{3.52}$$

At coinciding arguments $x = x'$ one has

$$B_t(j,x;j',x) = \sum_{l\le 0}\Big(\frac{t+x}{t-x}\Big)^{(j-j')/2} J_{j-l}\big(2\sqrt{t^2-x^2}\big)J_{j'-l}\big(2\sqrt{t^2-x^2}\big) \tag{3.53}$$

With these preparations for a general moment of the density field $\eta_t(j,x)$ one has the identity

$$\begin{aligned}
&\mathbb{E}_t\Big(\prod_{k=1}^m \eta_t(j_k,x_k)\Big) = \\
&= \lim_{M\to\infty}\lim_{\beta\to\infty} Z(\beta)^{-1}\mathrm{tr}\Big[e^{\beta\widehat{N}^M}e^{-t\widehat{H}_\mathrm{d}^M}\Big(\prod_{k=1}^m e^{-x_{\pi(k)}\widehat{H}_\mathrm{d}^M}a^*(j_{\pi(k)})a(j_{\pi(k)})e^{x_{\pi(k)}\widehat{H}_\mathrm{d}^M}\Big)e^{-t\widehat{H}_\mathrm{d}^M}\Big] \\
&= \det\Big(B_t(j_k,x_k;j_{k'},x_{k'})\Big)_{1\le k,k'\le m}.
\end{aligned} \tag{3.54}$$

As written, (3.54) is valid only for pairwise distinct $x_1,\dots,x_m$, where π is the unique permutation of $1,\dots,m$ such that the time ordering $-t \le x_{\pi(1)} < \dots < x_{\pi(m)} \le t$ is ensured. The spatial arguments $j_1,\dots,j_m$ are arbitrary. While each off-diagonal factor in the determinant has a jump discontinuity at $x_k = x_{k'}$, the determinant itself depends continuously on $x_1,\dots,x_m$, and thereby the continuous extension of (3.54) holds for all $j_k\in\mathbb{Z}$, $-t\le x_k\le t$, $k=1,\dots,m$.

As an application of (3.54) we establish the joint distribution of $\{\eta_t(j,x), j\in\mathbb{Z}\}$ for fixed x, $|x|\le t$. From (3.53) one derives immediately

$$e^{-xH_\mathrm{d}}B_te^{xH_\mathrm{d}} = g\,B_{\sqrt{t^2-x^2}}\,g^{-1} \tag{3.55}$$

where g is a multiplication operator with diagonal entries $g(n) = \big((t+x)/(t-x)\big)^{n/2}$. In (3.54) we take the limit of coinciding $x_k = x$, $k=1,\dots,m$, leaving $j_1,\dots,j_m$ pairwise distinct. Upon forming the determinant in (3.54) the similarity transformation g drops out and the result is (3.33) with B_t replaced by $B_{\sqrt{t^2-x^2}}$. Thus the joint distribution of $\{\eta_t(j,x), j\in\mathbb{Z}\}$ is again given through the discrete Bessel kernel with time parameter modified from t to $\sqrt{t^2-x^2}$.

The same conclusion can be drawn by taking the discrete PNG model as starting point. As explained in [6] the analogue of the fixed x distributions is given through the Krawtchouk polynomials. Their limit as $\delta\to 0$, $q = 4\delta^2$, $\delta\tau = t$, yields the joint distribution of $\{\eta_t(j,x), j\in\mathbb{Z}\}$ as given through (3.33) with parameter $\sqrt{t^2-x^2}$ instead of t.

In the following section we establish the scaling limit of the PNG droplet at locations of order $(yt^{2/3}, 2t+ut^{1/3})$, $y,u\in\mathbb{R}$. Since at $x = wt$, $|w|<1$, the distribution is determined by the discrete Bessel kernel $B_{t\sqrt{1-w^2}}$, we could instead of $w=0$ choose any other reference point $(wt, 2\sqrt{1-w^2}t)$ and relative displacements $(wt+yt^{2/3}, 2\sqrt{1-w^2}t+ut^{1/3})$. Except for scale factors, the limit $t\to\infty$ does not depend on the choice of w.

4 Edge scaling, convergence to the Airy process

We plan to establish that the statistics of the PNG droplet close to $x = 0$ converges to the Airy process. Since only moments are under control, the natural strategy is to prove that the Fermi field of Section 3 has a limit when viewed from the density edge. In particular, this implies, that the statistics of the last fermionic world line has a limit, which is the desired result.

Since $\langle h(0,t)\rangle = 2t$, the focus has to be at $x = 0 + yt^\alpha$, $j = 2t + ut^\beta$, $y, u \in \mathbb{R}$ fixed. Rescaling $\partial\psi/\partial x = H_t\psi$ accordingly one obtains

$$\frac{\partial}{\partial y}\psi = t^\alpha\big(- \psi(u + t^{-\beta}) - \psi(u - t^{-\beta}) + \frac{1}{t}(2t + ut^\beta)\psi(u)\big). \tag{4.1}$$

To have a limit operator as $t \to \infty$ requires

$$\alpha = \frac{2}{3}, \qquad \beta = \frac{1}{3}, \tag{4.2}$$

and with this choice (4.1) converges to

$$\frac{\partial}{\partial y}\psi = H\psi, \qquad H = -\frac{\partial^2}{\partial u^2} + u \tag{4.3}$$

regarded as a self-adjoint operator on $L^2(\mathbb{R})$. H is the Airy operator. The limit density field must correspond to free fermions with H as one-particle Hamiltonian. The fermions are in their ground state at zero chemical potential.

Let us first describe the Fermi field in more detail. The Airy operator H has $\mathbb{R}$ as spectrum, which is purely absolutely continuous. The generalized eigenfunctions are the Airy functions,

$$-\frac{d^2}{du^2}\mathrm{Ai}(u - \lambda) + u\mathrm{Ai}(u - \lambda) = \lambda\mathrm{Ai}(u - \lambda)\,. \tag{4.4}$$

In particular the completeness relation

$$\int d\lambda\, \mathrm{Ai}(u - \lambda)\mathrm{Ai}(v - \lambda) = \delta(u - v) \tag{4.5}$$

holds. K denotes the spectral projection onto $\{H \le 0\}$. Its integral kernel is the Airy kernel

$$\begin{aligned} K(u,v) &= \int_{-\infty}^0 d\lambda\, \mathrm{Ai}(u - \lambda)\mathrm{Ai}(v - \lambda) \\ &= \frac{1}{u - v}\big(\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)\big)\,. \end{aligned} \tag{4.6}$$

Next we introduce the Fermi field $a(u)$, $a^*(u)$, indexed by $u \in \mathbb{R}$. To distinguish from the fermions on a lattice we should use a different symbol. Since the latter will not reappear, we find it more convenient to stick to familiar notation. Integrated over a test function $f \in L^2(\mathbb{R})$ the Fermi field becomes $a(f) = \int du\, f^*(u)a(u)$, $a^*(f) = \int du\, f(u)a^*(u) = a(f)^*$. They satisfy the canonical anticommutation relations $\{a(f), a(g)\} = 0 = \{a^*(f), a^*(g)\}$ and $\{a(f), a^*(g)\} =$

(f,g) with $(\cdot,\cdot)$ denoting the inner product of $f \in L^2(\mathbb{R})$ [15], and generate the CAR algebra $\mathcal{A}$. On $\mathcal{A}$ we define the quasifree state ω through $\omega\big(a(f)\big) = 0 = \omega\big(a^*(f)\big)$ and $\omega\big(a^*(f)a(g)\big) = (f, K\,g)$. In particular, the moments of the density field are given by

$$\omega\Big(\prod_{n=1}^{m} a^*(u_k)a(u_k)\Big) = \det K(u_k,u_{k'})_{1\le k,k'\le m}, \tag{4.7}$$

for pairwise distinct $u_1,\ldots,u_m$, compare with (3.33). (4.7) is the m-th correlation function. It vanishes at coinciding points.

To extend to unequal times one defines the Euclidean propagator

$$\begin{aligned} K(u,y;u',y') &= \Big(e^{-yH}\big(K - 1\!\!1\,\theta(y-y')\big)e^{y'H}\Big)(u,u') \\ &= \operatorname{sign}(y'-y)\int d\lambda\theta\big(\lambda(y-y')\big)e^{\lambda(y'-y)}\mathrm{Ai}(u-\lambda)\mathrm{Ai}(v-\lambda), \end{aligned} \tag{4.8}$$

for $y \neq y'$, written in terms of eigenfunctions of H, and

$$K(u,y;u,y) = K(u,u), \tag{4.9}$$

compare with (3.47), (3.48). The quasifree state ω and the propagator are both determined by the Airy operator H, which implies that K depends only on $y-y'$.

The Airy field, denoted by $\xi(f,y) = \int du\, f(u)\xi(u,y)$, is the density field of the Fermi system defined through (4.8), (4.9). As in (3.54), its moments are of determinantal form and given by

$$\mathbb{E}\Big(\prod_{k=1}^{m}\xi(f_k,y_k)\Big) = \int\prod_{k=1}^{m} du_k\, f_k(u_k)\;\det\big(K(u_k,y_k;u_{k'},y_{k'})\big)_{1\le k,k'\le m}. \tag{4.10}$$

As it stands the left hand side of (4.10) is only defined for pairwise distinct $y_1,\ldots,y_m$, but as in (3.54) it can be continuously extended to arbitrary time arguments. Since K depends only on $y-y'$, the Airy field $\xi(f,y)$ is stationary in y.

Having introduced the limit object we turn to the edge scaling. Recall that the PNG droplet has curvature. Therefore we set the scaled density field of the multi-layer PNG model, denoted by ξ_t, as

$$\xi_t(u,y) = t^{1/3}\eta_t\big([2t + t^{1/3}(u-y^2)], t^{2/3}y\big)\,, \tag{4.11}$$

$[\,\cdot\,]$ denoting the integer part. When integrated over the real, smooth, and rapidly decreasing test function f we have

$$\begin{aligned} \xi_t(f,y) &= \int du f(u)\xi_t(u,y) \\ &= \sum_{j=-\infty}^{\infty} f\big(t^{-1/3}(j-2t)+y^2\big)\eta_t(j,t^{2/3}y) + \mathcal{O}(t^{-1/3}) \\ &= \sum_{\ell\le 0} f\big(t^{-1/3}(h_\ell(t^{2/3}y,t)-2t)+y^2\big) + \mathcal{O}(t^{-1/3})\,, \end{aligned} \tag{4.12}$$

where the error of order $t^{-1/3}$ results from integrating over cells of size $t^{-1/3}$ in the defining identity. Thus (4.12) shows that through controlling the limiting moments of $\xi_t(f,y)$ one can infer the limit of the scaled height lines $t^{-1/3}\big(h_\ell(t^{2/3}y,t)-2t\big)+y^2$.

Theorem 4.1 *Let $f_1,\ldots,f_m$ be smooth test functions of compact support. Then the following limit holds,*

$$\lim_{t\to\infty}\mathbb{E}_t\bigg(\prod_{k=1}^m \xi_t(f_k,y_k)\bigg)=\mathbb{E}\bigg(\prod_{k=1}^m \xi(f_k,y_k)\bigg). \tag{4.13}$$

Proof: Comparing (3.54) and (4.10) we have to establish that the propagator (3.47), properly scaled, converges to the continuum propagator (4.8). This limit can be handled most directly in the representation (3.52). We will need a separate argument for $y\neq y'$ and for the left, resp. right, limit $y=y'$.

The case $y<y'$ runs in complete parallel to $y>y'$. To simplify notation let us assume $y<y'$. The propagator for the scaled density field is

$$\begin{aligned}K_t(u,y;u',y') &= e^{-2t^{2/3}y}e^{(u-y^2)y}t^{1/3}B_t\big([2t+t^{1/3}(u-y^2)],t^{2/3}y,[2t+t^{1/3}(u'-y'^2)],t^{2/3}y'\big)\\ &\quad\times e^{2t^{2/3}y'}e^{-(u'-y'^2)y'}.\end{aligned} \tag{4.14}$$

y, y' are fixed and K_t is considered as a function on $\mathbb{R}^2$. It is constant over cells of size $t^{-1/3}$. We used here the freedom that the determinant of (3.54) does not change under a similarity transformation and multiplied with the factor $\exp\big(2t^{2/3}(y'-y)\big)\exp\big((u-y^2)y-(u'-y'^2)y'\big)$ which diverges as $t\to\infty$. K_t, with the obvious extension to $y>y'$, determines the moments of the scaled density field as

$$\mathbb{E}_t\bigg(\prod_{k=1}^m \xi_t(f_k,y_k)\bigg)=\int\prod_{k=1}^m du_k\, f_k(u_k)\ \det\big(K_t(u_k,y_k;u_{k'},y_{k'})\big)_{1\le k,k'\le m}+\mathcal{O}(t^{-1/3}), \tag{4.15}$$

clearly analoguous to (4.10).

We insert (3.52) into (4.14). Then

$$\begin{aligned}K_t(u,y;u',y') &= t^{-1/3}\sum_{l\in t^{-1/3}\mathbb{N}_-} e^{(y'-y)l}t^{1/3}J_{[2t+t^{1/3}(u-y^2-l)]}\big(2t\sqrt{1-t^{2/3}y^2}\big)\\ &\quad\times t^{1/3}J_{[2t+t^{1/3}(u'-y'^2-l)]}\big(2t\sqrt{1-t^{2/3}y'^2}\big)\\ &\quad\times\bigg\{\exp\big(-(2t^{2/3}y-(u-y^2)y-ly)\big)\bigg(\frac{1+t^{-1/3}y}{1-t^{-1/3}y}\bigg)^{\big(2t+t^{1/3}(u-y^2-l)\big)/2}\\ &\quad\times\exp\big((2t^{2/3}y'-(u-y'^2)y'-ly')\big)\bigg(\frac{1-t^{-1/3}y'}{1+t^{-1/3}y'}\bigg)^{\big(2t+t^{1/3}(u'-y'^2-l)\big)/2}\bigg\}\end{aligned} \tag{4.16}$$

If $y < y'$ and $l \leq 0$, the term $\{\cdots\}$ is uniformly bounded in t, l and converges to 1 as $t \to \infty$. In fact this holds uniformly in u, u' on compact sets. By a result of Landau [17]

$$\sup_n t^{1/3} |J_n(2t)| \leq c/2^{1/3} \tag{4.17}$$

with $c = 0.7857\cdots$. From the asymptotics of integer Bessel functions, cf. (9.3.23) of [14], we conclude, uniformly for u varying over a compact set,

$$\lim_{t\to\infty} J_{[2t+t^{1/3}(u-y^2)]}\big(2t\sqrt{1-t^{2/3}y^2}\big) = \mathrm{Ai}(u). \tag{4.18}$$

Since $e^{(y'-y)l}$, $l \leq 0$, is integrable, by dominated convergence

$$\lim_{t\to\infty} K_t(u, y; u', y') = K(u, y; u', y') \tag{4.19}$$

uniformly over compact u, u' sets.

Next we consider $y' \searrow y$ in (4.14), the right hand limit being handled analogously. If the discrete Bessel kernel is transformed according to (3.55), then

$$\begin{aligned} K_t(u, y; v, y) \quad &= \quad e^{y(u-u')} \left(\frac{1+t^{-1/3}y}{1-t^{-1/3}y}\right)^{[2t+t^{1/3}(u-y^2)]/2} \left(\frac{1-t^{-1/3}y}{1+t^{-1/3}y}\right)^{[2t+t^{1/3}(v-y^2)]/2} \\ &\quad \times t^{1/3} B_{t\sqrt{1-t^{-2/3}y^2}}\big([2t+t^{1/3}(u-y^2)], [2t+t^{1/3}(v-y^2)]\big). \end{aligned} \tag{4.20}$$

The first factor is uniformly bounded over compact u, v sets and converges to 1 as $t \to \infty$. By Proposition (4.1) of [18] the discrete Bessel kernel with our scaling converges to the Airy kernel $K(u, v)$ uniformly on compact u, v sets as $t \to \infty$.

We conclude that (4.19) holds not only for $y \neq y'$ but also for its right and left limits. Our claim follows by taking the limit $t \to \infty$ in (4.15) which then yields (4.10). □

The Airy field $\xi(f, y)$ is stationary in y. $\xi(f, y)$ is a point process for fixed y. Its average density is given by

$$\mathbb{E}(\xi(u, y)) = -u\,\mathrm{Ai}(u)^2 + \mathrm{Ai}'(u)^2 \tag{4.21}$$

which has the asymptotics [19]

$$\mathbb{E}\big(\xi(u, y)\big) \simeq \begin{cases} \frac{1}{\pi}|u|^{1/2} - \frac{1}{4\pi|u|}\cos(4|u|^{3/2}/3) + \mathcal{O}\big(|u|^{-5/2}\big) & \text{for } u \to -\infty, \\ \frac{17}{96\pi}u^{-1/2}\exp\big(-4u^{3/2}/3\big) & \text{for } u \to \infty. \end{cases} \tag{4.22}$$

Note that for $u \to \infty$ the density decays quickly because of the increasing linear potential, whereas for $u \to -\infty$ the density is limited through the Fermi exclusion. In particular, the point process for $\eta(f, y)$ has a last point at $h_0(y)$ with probability one. Since all points are distinct [12], one can label as

$$h_0(y) > h_{-1}(y) > \dots\ . \tag{4.23}$$

$y \mapsto h_\ell(y)$, $\ell \in \mathbb{N}_-$, are the fermionic world lines underlying the Airy field. As to be shown in Appendix A $y \mapsto h_\ell(y)$ is continuous with probability one. Moreover $\langle \big(h_\ell(y) - h_\ell(y')\big)^2 \rangle \simeq 2|y-y'|$, which suggests that the path measure for $\{h_\ell(y), |y| < c, \ell = -M, \dots, 0\}$ is absolutely continuous with respect to the Wiener measure, i.e. locally $h_\ell(y)$ is a modified Brownian motion.

Our main focus is the last fermion line.

Definition 4.2 *Let $\xi(f, y)$ be the Airy field. The last world line, $h_0(y)$, is called the Airy process and denoted by $A(y)$.*

We collect the basic properties of the Airy process.

Theorem 4.3 *The Airy process $A(y)$ has continuous sample paths. $A(y)$ is stationary. For given y, $A(y)$ has the distribution of χ_2 of Tracy-Widom, see below eq. (1.1).*

The convergence of the multi-layer PNG model to the Airy field implies that the shape fluctuations of the PNG droplet converge to the Airy process as $t \to \infty$. The following theorem is the precise version of the main result, Theorem 1.1, stated in the Introduction.

Theorem 4.4 *Let $h(x, t)$ be the height of the PNG droplet and $h_t(y)$ its scaled version according to (1.3). Let $A(y)$ be the Airy process. Then for any m, y_j, $a_j \in \mathbb{R}$, $j = 1, \dots, m$, we have*

$$\lim_{t\to\infty} \mathbb{P}_t\big(\{h_t(y_j) + y_j^2 \le a_j, j = 1, \dots, m\}\big) = \mathbb{P}\big(\{A(y_j) \le a_j, j = 1, \dots, m\}\big)\,. \tag{4.24}$$

Proof: Let f_j be the indicator of the interval (a_j, ∞). Then (4.24) means

$$\lim_{t\to\infty} \mathbb{P}_t\Big(\bigcap_{j=1}^{m}\{\xi_t(f_j, y_j) = 0\}\Big) = \mathbb{P}\Big(\bigcap_{j=1}^{m}\{\xi(f_j, y_j) = 0\}\Big)\,. \tag{4.25}$$

We choose a sufficiently large and split as $f_j = f_j^a + g^a$, where f_j^a is the indicator function of the interval $(a_j, a]$ and g^a is the one of (a, ∞). Then $\xi_t(f_j, y_j) = \xi_t(f_j^a, y_j) + \xi_t(g^a, y_j)$. By Theorem 4.1 the joint moments of $\xi_t(f_j^a, y_j)$, $j = 1, \dots, m$, converge to their limit. Since their limit measure on $\mathbb{R}^m$ is uniquely defined by its moments we conclude that (4.25) holds with f_j replaced by f_j^a. Up to constants the error term is bounded by a sum over terms of the form

$$\mathbb{P}_t\big(\xi_t(g^a, y_j) \ge 1\big) \le \mathbb{E}_t\big(\xi_t(g^a, y_j)\big) = \sum_{j \ge t^{1/3}a} B_{t\sqrt{1-t^{-2/3}y_j^2}}(j, j) \tag{4.26}$$

which has a bound $C(a)$ uniform in t such that $C(a) \to 0$ exponentially as $a \to \infty$, compare with (4.22). $\square$

5 Some properties of the Airy process, two-point function

The scale invariant statistics of the PNG droplet is governed by the Airy process. To gain some more quantitative information we have to study the Airy process, most prominently its distribution at a single point and at two points.

We denote by P_a the projection onto the interval (a,∞), i.e. $P_a\psi(u) = \chi_a(u)\psi(u)$ with $\chi_a(u)$ the indicator function of the set (a,∞). By definition

$$\mathbb{P}\big(\{A(y)\le a\}\big) = \mathbb{P}\big(\{\xi(\chi_a,y)=0\}\big)\,. \tag{5.1}$$

Let us set $\widehat{N}(\chi_a) = \int_a^\infty a^*(u)a(u)du$. Then

$$\mathbb{P}(\xi(\chi_a,y)=0) = \lim_{\beta\to\infty}\omega\big(e^{-\beta\widehat{N}(\chi_a)}\big). \tag{5.2}$$

Since ω is quasifree,

$$\omega\big(e^{-\beta\widehat{N}(\chi_a)}\big) = \det[1+(e^{-\beta}-1)P_aK], \tag{5.3}$$

where the determinant is in $L^2(\mathbb{R})$. P_aK is of trace class [4] and taking $\beta\to\infty$ yields

$$\mathbb{P}\big(\{A(y)\le a\}\big) = \det[1-P_aK]\,. \tag{5.4}$$

This determinant is studied in [4] and shown to be related to the Painlevé II differential equation. A plot for the probability distribution of $A(y)$ can be found, for example, in [2, 5].

The next quantity of interest is the joint distribution of $A(0)$, $A(y)$, where by reversibility it suffices to consider $y>0$. By the same scheme as before one computes

$$\begin{aligned}\mathbb{P}\big(\{A(0)\le a, A(y)\le b\}\big) &= \mathbb{P}\big(\{\xi(\chi_a,0)=0\}\cap\{\xi(\chi_b,y)=0\}\big)\\ &= \lim_{\beta\to\infty}\omega\big(e^{-\beta\widehat{N}(\chi_a)}e^{-y\widehat{H}}e^{-\beta\widehat{N}(\chi_b)}e^{y\widehat{H}}\big)\,.\end{aligned} \tag{5.5}$$

Since ω is quasifree,

$$\omega\big(e^{-\beta\widehat{N}(\chi_a)}e^{-y\widehat{H}}e^{-\beta\widehat{N}(\chi_b)}e^{y\widehat{H}}\big) = \det(1-B_\beta) \tag{5.6}$$

with

$$B_\beta = (1-e^{-\beta})(P_aK+e^{-yH}P_be^{yH}K) - (1-e^{-\beta})^2P_ae^{-yH}P_be^{yH}K. \tag{5.7}$$

$e^{-yH}P_be^{yH}K$ is trace class, cf. Appendix B. Thus

$$\mathbb{P}\big(\{A(0)\le a, A(y)\le b\}\big) = \det[1-B] \tag{5.8}$$

and

$$B = P_aK + e^{-yH}P_be^{yH}K - P_ae^{-yH}P_be^{yH}K. \tag{5.9}$$

Clearly, the determinant converges to 1 as $a,b\to\infty$ and to 0 as $a,b\to-\infty$.

The properties of (5.4) suggest that also the joint distribution might satisfy a differential equation. We did not succeed in finding one. Since the main interest is large y, we rely on standard asymptotics by employing the expansion

$$\log\det[1-B] = -\sum_{n=1}^\infty \frac{1}{n}\mathrm{tr}[B^n]\,. \tag{5.10}$$

When taking the trace of B^n, we see that $e^{yH}K$ is a bounded operator, but e^{-yH} remains unbalanced, since H is not bounded from below. E.g., $\mathrm{tr}[B]$ diverges as $\exp(y^{3/2})$ for $y \to \infty$. The form $\mathrm{tr}[B^n]$ is not suited for studying large y.

Such a situation is well known in the theory of Fermi systems [20, 21]. Since the Dirac sea is filled up to energy zero, one has to work with a new representation of the CAR algebra, which means to introduce field operators for the particles (energy ≥ 0) and for the holes (energy ≤ 0). In this representation the Hamiltonian is positive. There is no need here to enter into the full theory. It suffices to note that the series in (5.10) can be resummed such that only decaying exponentials appear. Let $\sigma_n = \big(\sigma(1), \ldots, \sigma(n)\big)$ be an n-letter word where each letter is either a or b, $\sigma_0 = \sigma_n$. Then

$$\log \mathbb{P}\big(\{A(0) \leq a, A(y) \leq b\}\big) = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\{\sigma_n\}} \mathrm{tr}\Big[\prod_{j=1}^{n} P_{\sigma(j)} K_{\sigma(j),\sigma(j+1)}\Big], \tag{5.11}$$

$\sigma(n+1) \equiv \sigma(1)$, with the following convention

$$\begin{aligned} K_{a,a} &= K_{b,b} = K \\ K_{a,b} &= e^{-yH}(K-1) \\ K_{b,a} &= e^{yH}K. \end{aligned} \tag{5.12}$$

The large y asymptotics is extracted from (5.11).

The spectral representation of $e^{-yH}(K-1)$, resp. $e^{yH}K$, yields an integral of the form $-\int_0^\infty d\lambda\, e^{-\lambda y} g_+(\lambda)$, resp. $\int_{-\infty}^0 d\lambda\, e^{\lambda y} g_-(\lambda)$, with some spectral functions g_+, g_-. For large y the weight concentrates at $\lambda = 0$ and results in the asymptotics $-g(0)/y$, resp. $g(0)/y$. Therefore a summand in (5.11) decays as $y^{-\alpha}$, where α is the number of broken bonds, i.e. $\ldots ab \ldots$ and $\ldots ba \ldots$, in the word σ_n of length n. The only words with no broken bonds are either all a's or all b's. They sum to

$$\det(1 - P_a K) \det(1 - P_b K). \tag{5.13}$$

As to be expected, the Airy process is mixing and far apart events become independent.

To order y^{-2} we only allow two broken bonds which for large y leads to

$$\begin{aligned} & y^{-2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m+n} \langle \mathrm{Ai}, (P_a K)^m P_a \mathrm{Ai} \rangle \langle \mathrm{Ai}, (P_b K)^n P_b \mathrm{Ai} \rangle \\ &= y^{-2} \int_0^\infty d\lambda\, \langle \mathrm{Ai}, P_a K (e^\lambda - P_a K)^{-1} P_a \mathrm{Ai} \rangle \langle \mathrm{Ai}, P_b K (e^\lambda - P_b K)^{-1} P_b \mathrm{Ai} \rangle\,, \end{aligned} \tag{5.14}$$

Here $\langle \mathrm{Ai}, \cdot\, \mathrm{Ai} \rangle$ means inner product in position space with respect to the Airy function $\mathrm{Ai}(u)$, i.e. for some operator R with integral kernel $R(u,v)$, $\langle \mathrm{Ai}, R\, \mathrm{Ai} \rangle = \int du \int dv \mathrm{Ai}(u) R(u,v) \mathrm{Ai}(v)$. To compute the two-point function to leading order in $1/y$ we integrate the probability measure with distribution function (5.8) against a and b and insert to leading order from (5.13), (5.14).

Using that $\det(1-P_aK)$ has a good decay for $a \to -\infty$ and $\langle \mathrm{Ai}, P_aK(e^\lambda - P_aK)^{-1}P_a\mathrm{Ai}\rangle$ has a good decay for $a \to \infty$, one is allowed to integrate by parts to obtain

$$\begin{aligned}&\langle A(0)A(y)\rangle - \langle A(0)\rangle\langle A(y)\rangle = \\ &\qquad \frac{1}{y^2}\int_0^\infty d\lambda \Big[\int da \det(1-P_aK)\langle \mathrm{Ai}, P_aK(e^\lambda - P_aK)^{-1}P_a\mathrm{Ai}\rangle\Big]^2 + \mathcal{O}(y^{-4}).\end{aligned} \tag{5.15}$$

The Airy process is positively correlated and has a slow decay as $1/y^2$.

6 Conclusions

The height statistics of the PNG droplet, for large t and under the scaling (1.3), are given by the Airy process. By universality such a result should be valid for any one-dimensional growth model in the KPZ class. The only condition is that locally the macroscopic shape must have a *non-zero curvature*. If the interface is flat on the average, other universal distributions will show up [2]. In such a situation, at present, no information on the multi-point statistics is available. For example in the PNG droplet we could lift the restriction that nucleation events are allowed only above the ground layer $[-t,t]$. By translation invariance of the dynamics and the initial condition, $h(x,0)=0$, we have $\langle h(x,t)\rangle = 2t$ for large t and the process $y \mapsto t^{-1/3}\big(h(yt^{2/3},t)-2t\big) = h_t^0(y)$ is stationary. For fixed y, $h_t^0(y)$ converges to the GOE Tracy-Widom distribution [5, 22]. The problem of the joint distribution of $h_t^0(y_1)$, $h_t^0(y_2)$ remains open.

The Airy process contains a wealth of statistical information, which cannot be resolved easily in Monte-Carlo simulations. In the standard numerical experiment one merely considers the second moment of the height differences. It is convenient to subtract the asymptotic mean as $\bar{h}(x,t) = h(x,t) - 2\sqrt{t^2-x^2}$. The quantity of interest is then

$$\langle\big(\bar{h}(x,t)-\bar{h}(0,t)\big)^2\rangle = G_t(x) \tag{6.1}$$

for large t. Our main result says that $G_t(x)$ is of scaling form and given by

$$G_t(x) \simeq t^{2/3}g(t^{-2/3}x). \tag{6.2}$$

The scaling function g can be expressed through the two-point function of the Airy process as

$$g(y) = \langle\big(A(y)-A(0)\big)^2\rangle. \tag{6.3}$$

For small y, one has

$$g(y) \simeq 2|y|, \quad y \to 0. \tag{6.4}$$

On the other hand, for large y, $A(y)$ becomes independent from $A(0)$ and

$$g(y) \simeq 2a_2, \qquad |y| \to \infty \tag{6.5}$$

with a_2 the truncated second moment of χ_2, $a_2 = \langle\chi_2^2\rangle - \langle\chi_2\rangle^2 \simeq 0.81320$ numerically. The asymptotics (5.15) says that

$$g(y) \simeq 2a_2 - c|y|^{-2}, \qquad |y| \to \infty, \tag{6.6}$$

to next order. Because of the inverse operator in (5.15) the positive constant c cannot be readily evaluated. A differential equation, like the Painlevé II for the single point distribution, would be helpful.

Appendix A: Continuous sample paths of the Airy process

Let $t \mapsto X(t)$ be a stochastic process with values in $\mathbb{R}$. By a criterion of Kolmogorov, cf. [23], Theorem 2.23, if for some constant c

$$\mathbb{E}(|X(t) - X(s)|^4) \le c|t-s|^2, \tag{A.1}$$

then $t \mapsto X(t)$ is continuous (in fact Hölder continuous with exponent $< \frac{1}{4}$) with probability one. Since (5.8) provides the joint distribution of $A(0)$, $A(y)$, (A.1) should be an easy exercise. We did not succeed and rely on a more indirect argument.

Let the test function f be smooth and of compact support. By the results of Section 4 we have

$$\xi(f,y) = \sum_{j\le 0} f\big(h_\ell(y)\big). \tag{A.2}$$

In the Lemma below we will prove that

$$\mathbb{E}\big((\xi(f,y) - \xi(f,y'))^4\big) \le c_f (y-y')^2. \tag{A.3}$$

Therefore $y \mapsto \xi(f,y)$ is continuous with probability one. Since the vague topology on locally finite point measures is countably generated, the trajectory $y \mapsto \sum_{\ell\le 0}\delta(h_\ell(y)-u)$ is continuous in the vague topology with probability one. The convergence of a sequence of locally finite point measures in the vague topology is equivalent to the convergence of each atom [24]. Thus (A.3) implies that $y \mapsto h_\ell(y)$ for each ℓ is continuous with probability one. In particular $A(y) = h_0(y)$ is continuous.

Lemma A.1 *Let f be smooth and of compact support. Then there is a constant c_f such that*

$$\mathbb{E}\big((\xi(f,y) - \xi(f,0))^4\big) \le c_f y^2. \tag{A.4}$$

Proof: As a warm-up, and to fix notation, we first compute the second moment. We suppress f and set $\xi(f,y) = \xi_y$. We take $y \ge 0$. $y \le 0$ follows from reversal symmetry $y \mapsto -y$ and stationarity in y. As shorthand we define $L = K - 1$. $-L$ is the projection operator onto $H \ge 0$. We regard f as a multiplication operator, $f\psi(u) = f(u)\psi(u)$, and formally set $f_y = e^{yH} f\, e^{-yH}$.

By construction, in the expressions below, only the bounded operators Ke^{yH} and $e^{-yH}L$ appear as factors. By the determinantal formula (4.10)

$$\begin{aligned}\mathbb{E}\big((\xi_y-\xi_0)^2\big) &= 2\big(\mathbb{E}(\xi_0^2)-\mathbb{E}(\xi_0\xi_y)\big)\\ &= -2\big(\mathrm{tr}(fKfL)-\mathrm{tr}(fKf_yL)\big)=-2\mathrm{tr}(fK(f-f_y)L).\end{aligned}\tag{A.5}$$

From the spectral representation, one concludes differentiability of any order in y, $y>0$, and up to linear order

$$\begin{aligned}\mathbb{E}\big((\xi_y-\xi_0)^2\big) &= 2y\,\mathrm{tr}(fK[H,f]L)+\mathcal{O}(y^2)\\ &= 2y\big(\mathrm{tr}(fK[H,f]K)-\mathrm{tr}(fK[H,f])\big)+\mathcal{O}(y^2).\end{aligned}\tag{A.6}$$

For real operators one has $\mathrm{tr}(AB)=\big(\mathrm{tr}(AB)\big)^*=\mathrm{tr}(B^*A^*)$. Since $[H,f]^*=-[H,f]$, the first summand vanishes and

$$\begin{aligned}\mathbb{E}\big((\xi_y-\xi_0)^2\big) &= y\,\mathrm{tr}(fK[H,f]L)+\mathcal{O}(y^2)=y\,\mathrm{tr}(K[f,[H,f]])+\mathcal{O}(y^2)\\ &= 2y\,\mathrm{tr}K(f')^2+\mathcal{O}(y^2).\end{aligned}\tag{A.7}$$

The variance (A.7) implies that $\mathbb{E}\big((h_\ell(y)-h_\ell(0))^2\big)=2|y|+\mathcal{O}(y^2)$ for each ℓ in the limit $y\to 0$, as to be expected from the construction of the Airy field.

The fourth moment requires more effort. We have, using stationarity and reversibility in y,

$$\mathbb{E}\big((\xi_y-\xi_0)^4\big)=2\big(\mathbb{E}(\xi_0^4)-4\mathbb{E}(\xi_0^3\xi_y)+3\mathbb{E}(\xi_0^2\xi_y^2)\big).\tag{A.8}$$

We again use the determinant formula (4.10), which most conveniently is decomposed into cycles. For the fourth moment there are 4! permutations. They subdivide into (i) four 1-cycles (1 term), (ii) two 1-cycles plus one 2-cycle (6 terms), (iii) two 2-cycles (3 terms), (iv) one 1-cycle plus one 3-cycle (8 terms), and (v) one 4-cycle (6 terms). The sign of the permutation will be of no significance for the argument.

(i) and (ii) vanish, (iii) is the "Gaussian" term,

$$\text{(iii)}=3\mathbb{E}\big((\xi_y-\xi_0)^2\big)^2\le c_f y^2\tag{A.9}$$

from (A.7). For (iv) we have a 1-cycle and two 3-cycles in reverse order. Summing over all such cycles yields

$$\begin{aligned}\text{(iv)} &= 12\mathrm{tr}(Kf)\big(-\mathrm{tr}(fKf_yLfL)+\mathrm{tr}(fKf_yLf_yL)\\ &\qquad -\mathrm{tr}(fKfKf_yL)+\mathrm{tr}(fKf_yKf_yL)\big).\end{aligned}\tag{A.10}$$

Since $\mathrm{tr}(fKf_yLfL)^*=\mathrm{tr}fKf_yLf_yL$ and $\mathrm{tr}(fKfKf_yL)^*=\mathrm{tr}fKf_yKf_yL$ the contribution (iv) vanishes.

It remains to study the 4-cycles. For their sum one obtains

$$\begin{aligned}
\text{(v)} \quad = \quad & \mathrm{tr}(fKfKfK(f-f_y)L) - 3\mathrm{tr}(fKfK(f-f_y)KfL) \\
& + \mathrm{tr}(fKfK(f-f_y)LfL) - 3\mathrm{tr}(fKfKfL(f-f_y)L) \\
& + \mathrm{tr}(fKfK(f-f_y)LfL) - 3\mathrm{tr}(fK(f-f_y)Kf_yLfL) \\
& + \mathrm{tr}(fKfLfK(f-f_y)L) - 3\mathrm{tr}(fK(f-f_y)LfKf_yL) \\
& + \mathrm{tr}(fK(f-f_y)LfKfL) - 3\mathrm{tr}(fKf_yLfK(f-f_y)L) \\
& + \mathrm{tr}(fK(f-f_y)LfLfL) - 3\mathrm{tr}(fKf_yL(f-f_y)LfL), \qquad \text{(A.11)}
\end{aligned}$$

where we combined the terms such that the order y is manifest. By the spectral theorem (A.11) is differentiable to any order in y, $y > 0$. Thus to complete the proof it suffices to show that the linear order of (A.11) vanishes. Since $f - f_y \simeq -y[H, f]$, we obtain

$$\begin{aligned}
\text{(v)} \quad = \quad & y\,\mathrm{tr}\big([H,f](LfKfKfK - 3KfLfKfK + 2LfLfKfK - 3LfKfKfL \\
& -3KfLfLfK - 4LfKfLfK + LfLfLfK - 3LfLfKfL)\big) + \mathcal{O}(y^2). \qquad \text{(A.12)}
\end{aligned}$$

We substitute $L = K - 1$ and use that $[H, f]^* = -[H, f]$. The term with four K's reads

$$-12\mathrm{tr}([H,f]KfKfKfK) = 0. \qquad \text{(A.13)}$$

The term with three K's reads

$$-6\mathrm{tr}\big([H,f]fKfKfK + Kf^2KfK + KfKf^2K + KfKfKf)\big) = 0, \qquad \text{(A.14)}$$

since first and fourth summand and second and third summand cancel. The term with two K's reads

$$-\mathrm{tr}\big([H,f](6fKfKf + 2Kf^3K + 3fKf^2K + 3Kf^2Kf)\big) = 0, \qquad \text{(A.15)}$$

since the first and second summand vanish and since the third summand cancels against the fourth one. The term with one K reads, upon adding the adjoint,

$$\mathrm{tr}([H,f]f^3 - f^3[H,f] - 3f[H,f]f^2 + 3f^2[H,f]f). \qquad \text{(A.16)}$$

At this point the specific structure of H enters. We have $[H, f] = -f'' - 2f'(d/du)$. Working out the commutators ensures also that the last term vanishes. Thus the linear order from the 4-cycles vanishes. The quadratic order does not vanish, however, and reflects the deviations from the local Brownian motion statistics. □

Appendix B: Trace class properties

As shown in [4] P_aK is of trace class. If one establishes that $R = e^{-yH}P_be^{yH}K$ is of trace class, then each summand in (5.9) is separately of trace class. In the energy representation R has the

kernel

$$R(\lambda', \lambda) = e^{-y\lambda'} \int_b^\infty dx\, \mathrm{Ai}(x - \lambda')\mathrm{Ai}(x - \lambda)e^{\lambda y}\chi(\lambda \leq 0). \tag{B.1}$$

If $\lambda \leq 0$, $\lambda' \geq 0$, the integral is bounded uniformly in λ, λ'. If $\lambda \leq 0$, $\lambda' \leq 0$, one uses that $\mathrm{Ai}(x - \lambda') \leq c\, e^{-\frac{2}{3}|x-\lambda'|^{3/2}}$ for large negative λ'. Thus the integral dominates the factor $e^{-\lambda' y}$. This implies the bound

$$|R(\lambda', \lambda)| \leq c\, e^{-\gamma(|\lambda'|+|\lambda|)} \tag{B.2}$$

with suitable constants $c, \gamma > 0$. Hence R is of trace class.

Note added in proof: In their recent preprint [25] Okounkov and Reshetikhin consider the $(1, 1, 1)$ interface of the three-dimensional Ising model at zero temperature, which maps onto a domino tiling of a form similar to the Aztec diamond. They prove that correlations are of determinantal form and compute the limit shape. Their Theorem 1 is the analogue of our Eq. (3.54).

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